International Journal of Theoretical Physics, Vol. 19, No. 11, 1980

# Modal Quantum Logic and Its Dialogic Foundation

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*Received March 24, 1980*

In this paper we investigate a modal calculus of quantum metalogic which is complete and sound with respect to a dialogic semantics. This calculus called $M_{\text{eff}}(Q_{\text{eff}})$ has three parts: one covering the formal metalogic, one reflecting the calculus of the object language, and one which is a link between object language and metalanguage. This third part is invariant with respect to a variation of the object language.

## 1. INTRODUCTION

In 1934 R. Carnap (1934) introduced the concept of *necessity* on the level of a metalanguage. O. Becker (1952) and later on P. Lorenzen (Lorenzen and Schwemmer, 1975) supplemented this idea using a "knowledge" $W$ by which one can define a proposition $A$ as necessary iff it can be deduced given any "knowledge" $W$. This approach uses a *dialog semantics* referring to the conditions under which propositions can be proved. The dialog semantics, first used to get an operational foundation of the calculus of the object language (Mittelstaedt, 1978; Lorenzen and Schwemmer, 1975), is also useful on the level of the metalanguage (Mittelstaedt, 1979). Furthermore a dialog semantics can be established for the modal metapropositions, as will be shown further.

The object language we will consider for the present is the language that must be used if one talks about quantum physical objects. The corresponding calculus is the *effective quantum logic* $Q_{\text{eff}}$ (Mittelstaedt, 1978, p. 96) extended by the "factual beginnings" $A_j \leqslant B_j$ which reflect the



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empirical laws of quantum theory (Mittelstaedt, 1979). The extended calculus is denoted by $Q_{eff}^{(f)}$. The restriction to the effective quantum logic in the following sections is not really important because our results are independent of the object language, as will be shown in Theorem 5.3.

An elementary proposition of our metalanguage is a statement "the figure $A \leqslant B$ is deducible in $Q_{eff}$" denoted by $\vdash_{\overline{Q_{eff}}} A \leqslant B$ or $A \lesssim B$. By this definition the elementary metapropositions are *proof definite* (cf. Mittelstaedt, 1978, p. 49). If $\vdash_{Q_{eff}^{(f)}} W \leqslant A$ is true $A$ is said to be "necessary with respect to $W$" and we will write $\Delta_W A$. Other modalities and their illustration in Hilbert space can be found in Mittelstaedt (1979). The proposition $W$ contains our knowledge of the physical system we are talking about, i.e., its preparation.

In Section 2 we build up compound metapropositions, e.g., $\overline{\bigwedge}_W \bar{A}^j(W \lesssim A_i, V \lesssim B_j)$ at which $\bar{A}^j$ is a connection of the elementary metapropositions $W \lesssim A_i$ and $V \lesssim B_j$ ($i = 1, \ldots, r$; $j = 1, \ldots, s$; $r, s \in \mathbb{N}$). The index $j$ at the $\bar{A}$ indicates that within $\bar{A}^j$ there are only meta*junctors* $\overline{\wedge}$, $\underline{\vee}$, $\Rightarrow$ and the metanegation $\urcorner$ but no metaquantifiers. (The more general case including quantifiers will not be considered here.) $\overline{\bigwedge}_W \bar{A}^j(W \lesssim A_i, V \lesssim B_j)$ is called a *modal* metaproposition; if such a metaproposition is true, I will call $\bar{A}^j(W \lesssim A_i, V \lesssim B_j)$ a *necessary* metaproposition [then I will write $\bar{A}^j(\Delta A_i, \vdash B_j)$] because the truth of $\bar{A}^j$ is independent of the special object proposition $W$.

But first of all we have to introduce a semantics for our metalanguage which gives us a concept of truth. This will be done in Sections 2 and 3 by considering dialogs about metapropositions. In Section 4 a tableaux calculus is given which is complete and sound with respect to the dialog semantics. In Section 5 we will consider other calculi $\tilde{M}_{eff}$ and $M_{eff}(Q_{eff})$ that are proved to be complete and sound with respect to the tableaux calculus and that allow us to deduce all necessary metapropositions. $M_{eff}(Q_{eff})$ will be called the "calculus of *modal quantum metalogic*."

There are two remarkable results:

(1) The calculus $M_{eff}(Q_{eff})$ consists of three parts: The first one reflects the object language, the second one yields the formally true metapropositions. The third one is a link between these two parts and is called the *modal part*, which makes it possible to use the "knowledge" of the object language on the level of the metalanguage; it is almost formally equivalent to the simplest axiomatic modal system $T$ (cf. Hughes and Cresswell, 1968; Burghardt, 1979, p. 89).

(2) The modal part of $M_{eff}(Q_{eff})$ is independent of the object language, i.e., using different object calculi $K$ we get modal metacalculi $M_{eff}(K)$ that are different only with respect to the part reflecting the object language.

## 2. THE MATERIAL AND THE SEMIFORMAL METADIALOG

The concept of a *dialog* can be introduced by *frame rules* (Stachow, 1978); furthermore some *argument rules* must be formulated in order to lay down the possibilities of attack and defense for compound propositions. The first argument rule, $A_f(1)$, can be understood as a definition of the connectives of the metalanguage (Mittelstaedt, 1979; Lorenz, 1968) (see Table 1). By this rule a compound metaproposition can be reduced within a dialog until an elementary metaproposition is reached. Then we need the next argument rule, $A_m(2)$ (see Table 2). An elementary metaproposition $\bar{A} \lesssim \bar{B}$ can be attacked by the question $\bar{A} \lesssim \bar{B}$? (in a *material* metadialog), and the defense $\bar{a}$! consists of a deduction of the figure $A \leqslant B$ in the calculus of the object language.

As in the object language (Stachow, 1976) we are only interested in finite dialogs. This restriction is performed by a third argument rule:

$A_f(3)$ a) $P$ is allowed to attack the same proposition of $O$ at most $n$ times.
b) $O$ is allowed to attack propositions of $P$ at most once. $O$ has to decide if he is going to defend against an attack by $P$ or if he is going to attack. In case he defends he no longer has the right to attack; in case he attacks the respective defense is no longer possible.

By the rules $A_f(1)$, $A_m(2)$, $A_f(3)$, and the frame rules the *material metadialog with bound n* is established. To characterize the *availability* of

**TABLE 1**

| $A_f(1)$ | | Connectives | Possibilities of attack | Possibilities of defense |
|---|---|---|---|---|
| | a) | $\bar{A} \bar{\wedge} \bar{B}$ | 1? <br> 2? | $\bar{A}$ <br> $\bar{B}$ |
| | b) | $\bar{A} \bar{\vee} \bar{B}$ | ? | $\bar{A}$ or $\bar{B}$ |
| | c) | $\bar{A} \Rightarrow \bar{B}$ | $\bar{A}$ | $\bar{B}$ |
| | d) | $\urcorner \bar{A}$ | $\bar{A}$ | |
| | e) | $\bar{\bigwedge}_x \bar{A}(x)$ | $n$ | $\bar{A}(n)$ |
| | f) | $\bar{\bigvee}_x \bar{A}(x)$ | ? | $\bar{A}(n)$ |

**TABLE 2**

| $A_m(2)$ | Elementary metaproposition | Possibility of attack | Possibility of defense |
|---|---|---|---|
| | $\bar{a}\ (=\bar{A} \lesssim \bar{B})$ | $\bar{a}$? | $\bar{a}$! |

$O$'s proposition an index (i) is assigned to them. According to rule $A_f(3)$ each proposition obtains the availability $n$ after it has been stated by $O$. If a proposition is attacked by $P$ the availability reduces by one. As usual (Stachow, 1976) we define a metaproposition $\bar{A}$ to be (materially) *true* iff $P$ has a strategy of success within a (material) dialog about $\bar{A}$.

There are compound metapropositions that can be defended successfully in a material dialog irrespective of the elementary metapropositions contained in it. These propositions are called *formally true*, the set of which can be covered by the calculus of *formal metalogic* (Mittelstaedt, 1979). But in this calculus it is not possible to use the knowledge of the object language, viz., the true object propositions. For instance, the metaproposition $V \lesssim A \Rightarrow V \lesssim A \vee B$ cannot be deduced in the formal metalogic. On the other hand we are not interested in the special form of the object proposition $A$ when we prove the truth of $V \lesssim A \Rightarrow V \lesssim A \vee B$, so the material dialog in which the opponent is obliged to defend $V \lesssim A$ by using the special form of $A$ is not useful (for our purpose). But if $O$ is dispensed with the defense of his elementary metapropositions—which of course can be taken over by the proponent without being obliged to defend this proposition—we must be careful. We have to exclude that $O$ asserts elementary metapropositions from which a contradiction, i.e., $V \lesssim \Lambda$, can be deduced. These considerations lead to the following argument rules establishing [together with the frame rules and $A_f(1)$ of course] the *semiformal metadialog* $D_s$:

$A_s(2)$a) Elementary metapropositions of $O$ are not attackable.

ba) Elementary metapropositions of $P$ which previously were asserted by $O$ are not attackable.

bb) In a position of a metadialog in which $P$ finally has asserted an elementary metaproposition $A \lesssim B$ (not yet asserted by $O$) $O$ can attack by asking $A \lesssim B$?. Let $\mathfrak{L} := \{A_i \lesssim B_i\}$ be the set of all elementary metapropositions previously asserted by $O$. The defense against $A \lesssim B$? is a deduction $A_1 \leqslant B_1, \ldots, A_r \leqslant B_r \vdash_{Q_{\text{eff}}} A \leqslant B$ in the calculus $Q_{\text{eff}}$ of the object language.

In $A_f(3)$ we add: $P$ is allowed to take over elementary metapropositions of $O$ at most $n$ times.

$A_s(4)$ In a position of a metadialog in which $O$ has asserted the elementary metapropositions $A_i \lesssim B_i$ $(i = 1, \ldots, r)$ $P$ is allowed to try to deduce $A_1 \leqslant B_1, \ldots, A_r \leqslant B_r \vdash_{Q_{\text{eff}}} V \leqslant \Lambda$. If he succeeds he has won the metadialog.

In $A_s(2)$bb) and $A_s(4)$ the syntactical completeness of $Q_{\text{eff}}$ (Stachow, 1978) is used by which it is possible to add a figure $A \leqslant B$ to the calculus instead of adding all figures appearing in a deduction of $A \leqslant B$.

## 3. THE SEMIFORMAL DIALOG FOR MODAL METAPROPOSITIONS

We specialize some rules of the semiformal metadialog for those cases in which a metadialog about modal metapropositions is performed.

*Step 1.* Every metadialog about a modal metaproposition begins as shown in Table 3. $P$ has a strategy of success for the modal metaproposition $\bar{\wedge}_W \bar{A}^j(W \lesssim A_i, V \lesssim B_j)$ iff he has a strategy of success for $\bar{A}^j(W_0 \lesssim A_i, V \lesssim B_i)$ without referring to the special form of $W_0$. So we can change the dialog rules in such a way that $P$ makes his arguments for any $W_0$. In order to do so we need a lemma, the simple proof of which will not be given here (Mittelstaedt, 1979; Burghardt, 1979). (We will write again $W$ instead of $W_0$.)

*Lemma 3.1. "Theorem of Aristotle":*

(a) $V \leqslant A_1, \ldots, V \leqslant A_m, \quad W \leqslant A_{m+1}, \ldots, W \leqslant A_p \vdash_{Q_{\text{eff}}} W_0 \leqslant B$ for all $W$
$\frown V \leqslant A_1, \ldots, V \leqslant A_m \vdash_{Q_{\text{eff}}} V \leqslant A_{m+1} \wedge \cdots \wedge A_p \to B$

(b) $V \leqslant A_1, \ldots, V \leqslant A_m \vdash_{Q_{\text{eff}}} W \leqslant B$ for all $W$
$\frown V \leqslant A_1, \ldots, V \leqslant A_m \vdash_{Q_{\text{eff}}} V \leqslant B.$

(c) $W \leqslant A_{m+1}, \ldots, W \leqslant A_p \vdash_{Q_{\text{eff}}} W \leqslant B$ for all $W$
$\frown \vdash_{Q_{\text{eff}}} V \leqslant A_{m+1} \wedge \cdots \wedge A_p \to B.$

*Lemma 3.2.*

(a) $V \leqslant A, V \leqslant B \vdash_{Q_{\text{eff}}} V \leqslant \Lambda \frown \vdash_{Q_{\text{eff}}} A \wedge B \leqslant \Lambda$

(b) $W \leqslant A, W \leqslant B \vdash_{Q_{\text{eff}}} W \leqslant \Lambda$ for all $W$
$\frown \vdash_{Q_{\text{eff}}} A \wedge B \leqslant \Lambda$

Using this lemma we get the following specialized form of the argument rules $A_s(2)$bb) and $A_s(4)$ [an exact reasoning for $A_s^m(4)$ can be found in Burghardt, 1979]:

$A_s^m(2)$bb) In a position of a metadialog in which $P$ finally has asserted the elementary metaproposition $W \lesssim B$ (not yet asserted by $O$) $O$ can attack by asking $W \lesssim B$?. Let $\mathfrak{L}$ be the set of all elementary metapropositions previously asserted by $O$. The defense against $W \lesssim B$? is a deduction

$\alpha$) $V \leqslant A_1, \ldots, V \leqslant A_m \vdash_{Q_{\text{eff}}} V \leqslant (A_{m+1} \wedge \cdots \wedge A_p \to B)$
if $\mathfrak{L} = \{V \lesssim A_1, \ldots, V \lesssim A_m, W \lesssim A_{m+1}, \ldots, W \leqslant A_p\}$,

**TABLE 3**

| | $O$ | $P$ |
|---|---|---|
| 0. | [ ] | $\bar{\wedge}_W \bar{A}^j(W \lesssim A_i, V \lesssim B_j)$ |
| 1. | $W_0$ | $\bar{A}^j(W_0 \lesssim A_i, V \lesssim B_j)$ |

β) $V \leqslant A_1, \ldots, V \leqslant A_m \vdash_{Q_{\text{eff}}} V \leqslant B$
if $\mathfrak{L} = \{V \lesssim A_1, \ldots, V \lesssim A_m\}$,
γ) $\vdash_{Q_{\text{eff}}} V \leqslant (A_{m+1} \wedge \cdots \wedge A_p \rightarrow B)$
if $\mathfrak{L} = \{W \lesssim A_{m+1}, \ldots, W \lesssim A_p\}$.

bc) If $P$ has asserted $V \lesssim B$, $O$ is allowed to attack by asking $V \lesssim B$?. The defense against this attack is a deduction of $V \leqslant B$ in $Q_{\text{eff}}$.

$A_s^m(4)$ In a position of a metadialog in which $O$ has asserted the elementary metapropositions $V \lesssim A_1, \ldots, V \lesssim A_m$, $W \lesssim A_{m+1}, \ldots, W \lesssim A_p$, $P$ is allowed to try to deduce $A_1 \wedge \cdots \wedge A_m \leqslant \Lambda$ or $A_{m+1} \wedge \cdots \wedge A_p \leqslant \Lambda$ in $Q_{\text{eff}}$. If he succeeds he wins the metadialog.

In a modal metaproposition $W$ represents a knowledge which shall not be false. This leads to the following rule:

$A^m(5)$ The player who asserts $W \lesssim \Lambda$ loses the metadialog.

*Step 2.* In the material and in the semiformal metadialog the defense of the elementary metaproposition $A \lesssim B$ has to be performed outside of the metadialog, viz., within the calculus $Q_{\text{eff}}$ of the object language. The deduction of the respective figure $A \leqslant B$ can be made by "reducing" $A \leqslant B$ to a beginning of $Q_{\text{eff}}$, i.e., the proponent uses the rules of $Q_{\text{eff}}$ in the opposite direction. If a rule of $Q_{\text{eff}}$ used by $P$ has two premises $O$ can choose the one by which $P$ has to continue. This procedure—let us call it a *material reduction*—is of course equivalent to the usual deduction starting with a beginning of the calculus. In order to translate this deduction into the metalanguage we introduce a calculus $\mathcal{K}$ which looks like $Q_{\text{eff}}$, but instead of $A \leqslant B$ in $Q_{\text{eff}}$ we write $V \lesssim A \rightarrow B$. Transforming all the rules $Q_{\text{eff}}(1.1)$–$Q_{\text{eff}}(5.3)$ (Mittelstaedt, 1978, p. 96) in this way we get the corresponding rules (K1.1)–(K5.3) on the level of the metalanguage. Of course $\mathcal{K}$ is complete and sound with respect to $Q_{\text{eff}}$, i.e.,

$$A_1 \leqslant B_1, \ldots, A_r \leqslant B_r \vdash_{Q_{\text{eff}}} A \leqslant B$$

$$\frown V \lesssim (A_1 \rightarrow B_1), \ldots, V \lesssim (A_r \rightarrow B_r) \vdash_{\mathcal{K}} V \lesssim (A \rightarrow B)$$

Furthermore this last assertion is equivalent to the metaproposition $V \lesssim (A_1 \rightarrow B_1) \barwedge \cdots \barwedge V \lesssim (A_r \rightarrow B_r) \Rightarrow V \lesssim (A \rightarrow B)$. So it is possible by means of $\mathcal{K}$ to replace the material reduction by a procedure only using metapropositions which we call "*formal reduction.*"

All steps necessary to defend a metaproposition can now be done on the level of the metalanguage and our argument rules can be formulated as follows:

$A^m_{sf}(2)$bb) ... The defense against the attack $W \lesssim B$? consists in asserting the new initial argument

α) $V \lesssim A_1 \curlywedge \cdots \curlywedge V \lesssim A_m \Rightarrow V \lesssim (A_{m+1} \wedge \cdots \wedge A_p \rightarrow B)$
if $\mathfrak{L} = \{V \lesssim A_1, \ldots, V \lesssim A_m, W \lesssim A_{m+1}, \ldots, W \lesssim A_p\}$,

β) $V \lesssim A_1 \curlywedge \cdots \curlywedge V \lesssim A_m \Rightarrow V \lesssim B$
if $\mathfrak{L} = \{V \lesssim A_1, \ldots, V \lesssim A_m\}$,

γ) $V \lesssim (A_{m+1} \wedge \cdots \wedge A_p \rightarrow B)$
if $\mathfrak{L} = \{W \lesssim A_{m+1}, \ldots, W \lesssim A_p\}$.

bc) In a position of a metadialog in which $P$ finally has asserted the elementary metaproposition $V \lesssim B$ (not yet asserted by $O$) $O$ can attack by asking $V \lesssim B$?. If $V \lesssim B$ is a premise of a rule of $\mathcal{K}$, $P$ can defend against $V \lesssim B$? by asserting a premise of this rule. If there are two premises $O$ can choose the one by which $P$ must continue the metadialog. If (K1.2) shall be used $P$ can choose a proposition $B$, and then $O$ chooses one of the premises.

c) $P$ is allowed to defend by bc) at most $n$ times running.

$A^m_{sf}(4)$ In a position of a metadialog in which $O$ has asserted the elementary metapropositions $V \lesssim A_1, \ldots, V \lesssim A_m$, $W \lesssim A_{m+1}, \ldots, W \lesssim A_p$, $P$ is allowed to continue by asserting the new initial argument $V \lesssim (A_1 \wedge \cdots \wedge A_m \rightarrow \Lambda)$ or $V \lesssim (A_{m+1} \wedge \cdots \wedge A_p \rightarrow \Lambda)$.

The rule $A^m_{sf}(2)$c) has been added for we are interested only in finite dialogs. This rule is taken into consideration by providing the arguments of $P$ with an availability index $(n)$. If $O$ attacks an elementary metaproposition $V \lesssim B^{(\nu)}$ $(\nu \in \{1, \ldots, n\})$ and $P$ defends using the rule $A^m_{sf}(2)$bc) then the new argument of $P$ gets the index $(\nu - 1)$.

The argument rules $A_f(1)$, $A_s(2)$a)–ba), $A^m_{sf}(2)$bb)–c), $A_f(3)$, $A^m_{sf}(4)$, $A^m(5)$ and the frame rules define the "*semiformal dialog for modal metapropositions with formal reduction*" $D^m_{sf}$ which covers a special domain of the semiformal metadialog $D_s$. As mentioned above we have the following equivalence:

$P$ has a strategy of success in $D_s$ for the modal metaproposition $\curlywedge_W \bar{A}^j(W \lesssim A_i, V \lesssim B_j)$ iff $P$ has a strategy of success in $D^m_{sf}$ for the metaproposition $\bar{A}^j(W \lesssim A_i, V \lesssim B_j)$.

The dialog $D^m_{sf}$ is our starting point for finding a calculus by which all true modal metapropositions can be deduced. This will be done in the proceeding sections.

## 4. THE TABLEAUX CALCULUS OF MODAL QUANTUM METALOGIC

**4.1. The Tableaux Calculus $\phi_n(D^m_{sf})$.** Stachow (1976) introduced the formal representation of a dialog game and the concept of *reduced positions* of a dialog. *Tableaux* represent certain reduced positions by means of the bijective mapping

$$\begin{array}{llcll} & |\overline{A}^{(\nu)} & \rightarrow & & \|\overline{A}^{(\nu)} \\ \bar{\pi}^{(\alpha)} & |\overline{A}^{(\nu)} & \rightarrow & \bar{\pi}^{(\alpha)} & \|\overline{A}^{(\nu)} \\ \bar{\pi}^{(\alpha)} & |\Gamma & \rightarrow & \bar{\pi}^{(\alpha)} & \|\Gamma \end{array}$$

where $\bar{\pi}$ denotes a system of metapropositions of $O$'s, $(\alpha)$ designates a respective system of availabilities, and furthermore $\Gamma$ designates a potential argument.

The rules (1.1), (2.1)–(2.4), (3.1)–(3.4), (5.1)–(5.2) of $\phi_n(D^m_{sf})$ establish the well-known intuitionistic tableaux calculus [cf. Lorenz, 1968; for the rules 6.1–6.2 in Lorenz, 1968, we use the notation (5.1)–(5.2)], e.g.,

$$(2.1) \qquad \bar{\pi}^{(\alpha)}\|\left[\overline{A}\right] \quad \text{and} \quad \bar{\pi}^{(\alpha)}\|\left[\overline{B}\right] \Rightarrow \bar{\pi}^{(\alpha)}\|\overline{A}\,\overline{\wedge}\,\overline{B}^{(n)}$$

The rules of the calculus $\mathcal{K}$ are covered by the corresponding tableaux (k1.1)–(k5.3), e.g.,

$$(\text{k}1.1) \Rightarrow \bar{\pi}^{(\alpha)}\|V \lesssim (A \rightarrow A)^{(\nu)}$$

Furthermore we have the following rules:

$$(2.5.1) \qquad \|V \lesssim A_1 \overline{\wedge} \cdots \overline{\wedge} V \lesssim A_m \Rightarrow V \lesssim (A_{m+1} \wedge \cdots \wedge A_p \rightarrow B)^{(n)}$$

$$\begin{array}{rl|l} \Rightarrow & \bar{\pi}^{(\alpha)} & \\ & V \lesssim A_1{}^{(n)} & \\ & \vdots & \\ & V \lesssim A_m{}^{(n)} & \\ & W \lesssim A_{m+1}^{(n)} & \\ & \vdots & \\ & W \lesssim A_p{}^{(n)} & W \lesssim B^{(n)} \end{array}$$

$$(2.5.2)\qquad \| V\lesssim A_1 \barwedge \cdots \barwedge V\lesssim A_m \Rightarrow V\lesssim B^{(n)} \Rightarrow \begin{array}{c||l} \overline{\text{Л}}^{(\alpha)} & \\ V\lesssim A_1{}^{(n)} & \\ \vdots & \\ V\lesssim A_m{}^{(n)} & W\lesssim B^{(n)} \end{array}$$

$$(2.5.3)\qquad \| V\lesssim (A_{m+1}\wedge \cdots \wedge A_p \to B)^{(n)} \Rightarrow \begin{array}{c||l} \overline{\text{Л}}^{(\alpha)} & \\ W\lesssim A_{m+1}^{(n)} & \\ \vdots & \\ W\lesssim A_p{}^{(n)} & W\lesssim B^{(n)} \end{array}$$

$$(5.3.1)\qquad \| V\lesssim (A_1\wedge \cdots \wedge A_m \to \Lambda)^{(n)} \Rightarrow \begin{array}{c||l} \overline{\text{Л}}^{(\alpha)} & \\ V\lesssim A_1{}^{(n)} & \\ \vdots & \\ V\lesssim A_m{}^{(n)} & \Gamma \end{array}$$

$$(5.3.2)\qquad \| V\lesssim (A_{m+1}\wedge \cdots \wedge A_p \to \Lambda)^{(n)} \Rightarrow \begin{array}{c||l} \overline{\text{Л}}^{(\alpha)} & \\ W\lesssim A_{m+1}^{(n)} & \\ \vdots & \\ W\lesssim A_p{}^{(n)} & \Gamma \end{array}$$

$$(6)\qquad \Rightarrow \begin{array}{c||l} \overline{\text{Л}}^{(\alpha)} & \\ W\lesssim \Lambda^{(n)} & \Gamma \end{array}$$

*Theorem 4.1. Completeness and soundness of* $\phi_n(D_{sf}^m)$ *with respect to the dialog game*. The metaproposition $\overline{A}^j(W\lesssim A_i, V\lesssim B_j) =: \overline{A}$ is true in the metadialog $D_{sf}^m$ with bound $n$ iff there is a $\nu \in \{1, \ldots, n\}$ so that the tableau $\| \overline{A}^{(\nu)}$ can be deduced in $\phi_n(D_{sf}^m)$.

*Proof.* For the completeness proof (which is performed by means of and induction) we show that every tableau corresponding to a reduced position of success is deducible in $\phi_n(D^m_{sf})$. We have the following possibilities:

(1) A position of success is a final position of success, i.e., one of the following positions:

(i) $\begin{array}{c|c} Л^{(\alpha)} & \\ \bar{a}^{(\nu)} & \bar{a}^{(n)} \end{array}$ Because of the rule $A^m_{sf}$(2)ba) $O$ is not allowed to attack and therefore $O$ has no subsequent move. The respective tableau is the deducible beginning (1.1) in $\phi_n(D^m_{sf})$.

(ii) $\begin{array}{c|c} \bar{Л}^{(\alpha)} & \\ W \lesssim \Lambda^{(n)} & \Gamma \end{array}$ $O$ loses the dialog because of the rule $A^m(5)$. The respective tableau is the deducible beginning (6) in $\phi_n(D^m_{sf})$.

(iii) $\bar{Л}^{(\alpha)} \mid V \lesssim A \to A^{(\nu)}$

$\vdots \quad \vdots$

(ix) $\bar{Л}^{(\alpha)} \mid V \lesssim (A \wedge \neg A \to \Lambda)^{(\nu)}$

$\nu \in \{1, \ldots, n\}$

$P$ asserts an elementary metaproposition which cannot be attacked by $O$ because of the rule $A^m_{sf}$(2)bd). The respective tableaux are the deducible beginnings within the rules (k1.1) –(k5.3).

(2) A position of success is a member of the move class of $P$, i.e.,

$$\begin{array}{l|l} \bar{Л}^{(\alpha)} & \\ V \lesssim A_1{}^{(n)} & \\ \vdots & \\ V \lesssim A_m{}^{(n)} & \\ W \lesssim A_{m+1}{}^{(n)} & \\ \vdots & \\ W \lesssim A_p{}^{(n)} & \Gamma \end{array} \quad =: \quad \bar{\mathfrak{L}}^{(\beta)} \mid \Gamma$$

Then at least one of the subsequent positions must be a position of success.

(2.1) In case $P$ defends one has to distinguish the following cases:

(i) $\bar{\mathfrak{L}}^{(\beta)} \mid [\bar{A}]$ The successors are $\mid V \lesssim (A_1 \wedge \cdots \wedge A_m \to \Lambda)^{(n)}$, $\mid V \lesssim (A_{m+1} \wedge \cdots \wedge A_p \to \Lambda)^{(n)}$, and $\bar{\mathfrak{L}}^{(\beta)} \mid \bar{A}^{(n)}$. One of these must be a position of success. The respective tableaux are the premises of the rules (5.1) and (5.3). By assumption (within our induction) the tableau related to the position of success is deducible in $\phi_n(D^m_{sf})$, and so is the conclusion of the respective rule, viz., $\bar{\mathfrak{L}}^{(\beta)} \| [\bar{A}]$, deducible too.

(ii) $\overline{\bar{\mathfrak{L}}^{(\beta)} \mid [\bar{A}, \bar{B}]}$ The successors are $\overline{\bar{\mathfrak{L}}^{(\beta)} \mid \bar{A}^{(n)}}$, $\overline{\mid V \lesssim (A_1 \wedge \cdots \wedge A_m \rightarrow \Lambda)^{(n)}}$, $\overline{\mid V \lesssim (A_{m+1} \wedge \cdots \wedge A_p \rightarrow \Lambda)^{(n)}}$, and $\overline{\bar{\mathfrak{L}}^{(\beta)} \mid \bar{B}^{(n)}}$. By the same arguments as above one gets the deducibility of $\bar{\mathfrak{L}}^{(\beta)} \| [\bar{A}, \bar{B}]$ using the rules (5.2) and (5.3).

(iii) $\overline{\bar{\mathfrak{L}}^{(\beta)} \mid [\ \ ]}$ The only successors are $\overline{\mid V \lesssim (A_1 \wedge \cdots \wedge A_m \rightarrow \Lambda)^{(n)}}$ and $\overline{\mid V \lesssim (A_{m+1} \wedge \cdots \wedge A_p \rightarrow \Lambda)^{(n)}}$. Using the rules (5.3) one gets the deducibility of the tableau $\bar{\mathfrak{L}}^{(\beta)} \| [\ \ ]$.

(2.2) In case $P$ attacks cf. Lorenz (1968).

(3) A position of success is a member of the move class of $O$, i.e.,

$$
\begin{array}{c|c}
\bar{\pi}^{(\alpha)} & \\
V \lesssim A_1{}^{(n)} & \\
\vdots & \\
V \lesssim A_m{}^{(n)} & \\
W \lesssim A_{m+1}^{(n)} & \\
\vdots & \\
W \lesssim A_p{}^{(n)} & \bar{C}^{(n)}
\end{array}
\quad =: \quad \overline{\bar{\mathfrak{L}}^{(\beta)} \mid \bar{C}^{(n)}}
$$

which is not a final position. Then all subsequent positions must be positions of success.

(3.1) $\bar{C}$ is a compound metaproposition $\bar{A} \overline{\wedge} \bar{B}$ $\bar{A} \underline{\vee} \bar{B}$, $\bar{A} \Rightarrow \bar{B}$, $\urcorner \bar{A}$, or the elementary metaproposition $W \lesssim B$ which does not occur in the left column $\bar{\mathfrak{L}}^{(\beta)}$. The successors corresponding to deducible tableaux (by assumption within our induction) are positions the tableaux of which are the premises of the rules (2.1)–(2.5). Using one of these rules one gets the tableau $\bar{\mathfrak{L}}^{(\beta)} \| \bar{C}^{(n)}$.

(3.2) $\bar{C}$ is the elementary metaproposition $V \lesssim B$. Then $V \leqslant B$ must be deducible in $Q_{\text{eff}}$ and it is not a beginning (otherwise the position $\overline{\bar{\mathfrak{L}}^{(\beta)} \mid \bar{C}^{(n)}}$ would be a final position). Because of the dialog rule $A^m_{sf}(2)$bc) the successors are the positions corresponding to the premises of the rules (k2.1), etc. These tableaux are deducible by assumption (for they are related to positions of success), and so is the conclusion $\bar{\mathfrak{L}}^{(\beta)} \| V \lesssim B^{(n)}$ of the respective rule deducible too.

For the soundness proof we show that every tableau deducible in $\phi_n(D_{sf}^m)$ corresponds to a position of success in the dialog game:

(1) All the beginnings in $\phi_n(D_{sf}^m)$ correspond to positions of success.

(2) Ad rules (2.1)–(2.4), (3.1)–(3.4), and (5.1)–(5.2) cf. Lorenz (1968). In the following I use the words *premise–position* and *conclusion–position* for a position corresponding to a premise or to a conclusion, respectively, in a rule. Using this definition we have to show that all conclusion–positions are positions of success.

(3) Ad rules (2.5.1)–(2.5.3): If $W \lesssim B$ occurs in the left column of the conclusion–position this position is a final position of success. Otherwise the premise–position is the only possible successor. By assumption the premise–positions are positions of success. Therefore all successors of the conclusion–position are positions of success, and the conclusion–position must be a position of success too.

(4) Ad rule (5.3): The respective conclusion–position is a member of the move class of $P$. In order to be a position of success at least one successor must be a position of success. The respective premise–positions are indeed successors and therefore positions of success by assumption.

(5) Ad rules (k2.1), etc.: Let us consider a certain rule; the conclusion–position $\overline{\pi^{(\alpha)} \mid V \lesssim D^{(\nu)}}$ shall not be a final position (therefore $\nu \neq 1$). Hence the successors are the premise–positions of certain rules of (k2.1), etc. These positions are positions of success by assumption in our induction, and therefore the conclusion–position must be a position of success too. ■

**4.2. The Effective Tableaux Calculus $\phi_{\text{eff}}(D_{sf}^m)$.** In order to catch all modal metapropositions which are true in the semiformal dialog $D_{sf}^m$ irrespective of a special bound we have to take the union of all dialogs with bound $n = 1,2,\ldots$ and, respectively, the union of all tableaux calculi $\phi_n(D_{sf}^m)$. The result is the *effective tableau calculus* $\phi_{\text{eff}}(D_{sf}^m)$ which looks like $\phi_n(D_{sf}^m)$ except that availability indices appear not any longer.

> *Theorem 4.2. Completeness and soundness of $\phi_{\text{eff}}(D_{sf}^m)$ with respect to $\cup_{n=1}^{\infty} \phi_n(D_{sf}^m)$.* A tableau $\| \overline{A}$ can be deduced in the calculus $\phi_{\text{eff}}(D_{sf}^m)$ iff there are an $n \in \mathbb{N}$ and a $\nu \in \{1,\ldots,n\}$ so that the tableau $\| \overline{A}^{(\nu)}$ can be deduced in $\phi_n(D_{sf}^m)$.

The simple proof is analogous to the one in Lorenz (1968).

## 5. THE PROPOSITIONAL CALCULUS OF MODAL QUANTUM METALOGIC

**5.1. The Calculus $\tilde{M}_{\text{eff}}$.** The figures $\overline{A} \leqq \overline{B}$ of the calculus $\tilde{M}_{\text{eff}}$ correspond to metaimplications $\overline{A} \Rightarrow \overline{B}$. The rules of this calculus can be handled easier than those of a tableaux calculus; therefore the modal metapropositions which are true with respect to the semiformal dialog $D_{sf}^{m}$ can be deduced in a (relative) very simple way. The *figures* deducible in $\tilde{M}_{\text{eff}}$ are expressions of the form $\bar{A} \leqq \bar{B}$ at which $\bar{A}$ and $\bar{B}$ are metapropositions $\bar{A}^{j}(W \lesssim A_i, V \lesssim B_j)$ with object propositions $A_i$ and $B_j$, the constant $V$ of the object language (Mittelstaedt, 1979, p. 89), and a variable $W$ for an object proposition. Furthermore $\bar{A}$ and $\bar{B}$ maybe are the *metaverum* $\underline{\mathrm{V}}$ (or the *metafalsum* $\overline{\Lambda}$) defined by: $\bar{A} \leqslant \underline{\mathrm{V}}$ (or $\overline{\Lambda} \leqslant \bar{A}$, respectively) for all metapropositions $\bar{A}$.

The *rules* of $\tilde{M}_{\text{eff}}$ including the *beginnings* consist of expressions $\bar{A} \leqq \bar{B}$ at which $\bar{A}$ and $\bar{B}$ either are variables for a metaproposition $\bar{A}^{j}(W \lesssim A_i, V \lesssim B_j)$ with fixed propositions $A_i$ and $B_j$ or have the form $\bar{A}^{j}(W \lesssim A_i, V \lesssim B_j)$ with variables $A_i$, $B_j$ for propositions of the object language. In any case, performing a deduction in $\tilde{M}_{\text{eff}}$ the place of the variable $W$ is not occupied by a fixed proposition. Remember that we are interested only in modal metapropositions which are true for *all* $W$!

The rules (1.1)–(5.2) of $\tilde{M}_{\text{eff}}$ are the well-known intuitionistic logic (Stachow, 1978), e.g.,

(1.1) $\Rightarrow \quad \bar{A} \leqq \bar{A}$

(1.2) $\bar{A} \leqq \bar{B}$ and $\bar{B} \leqq \bar{C} \Rightarrow \bar{A} \leqq \bar{C}$

The calculus $\mathcal{K}$ is reflected within $\tilde{M}_{\text{eff}}$ by the rules (M1.1)–(M5.3), e.g.,

(M1.1) $\Rightarrow \quad \underline{\mathrm{V}} \leqq V \lesssim (A \rightarrow A)$

(M1.2) $\underline{\mathrm{V}} \leqq V \lesssim (A \rightarrow B)$ and $\underline{\mathrm{V}} \leqq V \lesssim (B \rightarrow C) \Rightarrow \underline{\mathrm{V}} \leqq V \lesssim (A \rightarrow C)$

Furthermore we have the rules

(M0.1) $\Rightarrow \quad W \lesssim \Lambda \leqq \overline{\Lambda}$

(M0.3) $\Rightarrow \quad W \lesssim (A \rightarrow B) \leqq W \lesssim A \Rightarrow W \lesssim B$

(M0.5) $\Rightarrow \quad W \lesssim A \,\overline{\wedge}\, W \lesssim B \leqq W \lesssim (A \wedge B)$

(M0.11) $\Rightarrow \quad V \lesssim A \leqq W \lesssim A$

(M0.13) $\Rightarrow \quad V \lesssim (A \rightarrow B) \leqq V \lesssim A \Rightarrow V \lesssim B$

(M0.15) $\Rightarrow \quad V \lesssim A \,\overline{\wedge}\, V \lesssim B \leqq V \lesssim (A \wedge B)$

*Lemma 5.1.* The following rules are deducible in $\tilde{M}_{\text{eff}}$:

(R) $\quad \bar{A} \leqq \bar{B} \Leftrightarrow \underline{\mathrm{V}} \leqq \bar{A} \Rightarrow \bar{B}$

(4.2′) $\quad \bar{C} \leqq \bar{A} \Rightarrow \bar{B} \Rightarrow \bar{A} \,\overline{\wedge}\, \bar{C} \leqq \bar{B}$

(M0.7) $\quad \underline{\mathrm{V}} \leqq W \lesssim (A \rightarrow B) \Rightarrow \underline{\mathrm{V}} \leqq (W \lesssim A \Rightarrow W \lesssim B)$

(M0.17) $\quad \underline{\mathrm{V}} \leqq V \lesssim (A \rightarrow B) \Rightarrow \underline{\mathrm{V}} \leqq (V \lesssim A \Rightarrow V \lesssim B)$

*Proof.*

ad (R): "⇒" $\bar{A}\leqq\bar{B}\Rightarrow\bar{A}\bar{\wedge}\underline{\mathrm{V}}\leqq\bar{B}$ (1.1), (2.1), (1.2)

$\Rightarrow\underline{\mathrm{V}}\leqq\bar{A}\Rightarrow\bar{B}$ (4.2)

"⇐" $\underline{\mathrm{V}}\leqq\bar{A}\Rightarrow\bar{B}\Rightarrow\bar{A}\bar{\wedge}\underline{\mathrm{V}}\leqq\bar{B}$ (4.2′) (1)

$\bar{A}\leqq\underline{\mathrm{V}}$ (definition of $\underline{\mathrm{V}}$) $\Rightarrow\bar{A}\leqq\bar{A}\bar{\wedge}\underline{\mathrm{V}}$ (1.1), (2.3) (2)

(1), (2), (1.2) $\Rightarrow\bar{A}\leqq\bar{B}$

ad (4.2′): premise$\Rightarrow\bar{A}\bar{\wedge}\bar{C}\leqq\bar{A}\Rightarrow\bar{B}$ (2.2), (1.2)

$\Rightarrow\bar{A}\bar{\wedge}\bar{C}\leqq\bar{A}\bar{\wedge}(\bar{A}\Rightarrow\bar{B})$ (1.1), (2.3), (1.2)

$\Rightarrow\bar{A}\bar{\wedge}\bar{C}\leqq\bar{B}$ (4.1), (1.2)

ad (M0.7): Use (M0.3) and (1.2).

ad (M0.17): Use (M0.13) and (1.2). ■

*Theorem 5.1. Completeness and soundness of $\tilde{M}_{\mathrm{eff}}$ with respect to $\phi_{\mathrm{eff}}(D_{sf}^{m})$.* A tableau $\|\bar{A}$ can be deduced in $\phi_{\mathrm{eff}}(D_{sf}^{m})$ iff the figure $\underline{\mathrm{V}}\leqq\bar{A}$ can be deduced in $\tilde{M}_{\mathrm{eff}}$.

*Proof.* For the completeness proof we consider the following mapping:

$$g_3:\quad \{\text{tableaux of } \phi_{\mathrm{eff}}(D_{sf}^{m})\}\rightarrow\{\text{figures of } \tilde{M}_{\mathrm{eff}}\}$$

| | | |
|---|---|---|
| $\bar{\pi}\|\bar{A}$ | $\rightarrow$ | $\hat{\pi}\leqq\bar{A}$ |
| $\bar{\pi}\|[\ \ ]$ | $\rightarrow$ | $\hat{\pi}\leqq\bar{\wedge}$ |
| $\|\bar{A}$ | $\rightarrow$ | $\underline{\mathrm{V}}\leqq\bar{A}$ |
| $\bar{\pi}\|[\bar{A},\bar{B}]$ | $\rightarrow$ | $\hat{\pi}\leqq\bar{A}\underline{\vee}\bar{B}$ |
| $\bar{\pi}\|[\bar{A}]$ | $\rightarrow$ | $\hat{\pi}\leqq\bar{A}$ |

We have to prove that all the rules of $\phi_{\mathrm{eff}}(D_{sf}^{m})$ transformed by means of $g_3$ are deducible in $\tilde{M}_{\mathrm{eff}}$.

(1) ad (1.1)–(5.2.2): cf. Stachow (1978).
(2) ad (2.5.1): Proposition: The rule

$$\underline{\mathrm{V}}\leqq V\lesssim A_1\bar{\wedge}\cdots\bar{\wedge}V\lesssim A_m\Rightarrow V\lesssim(A_{m+1}\wedge\cdots\wedge A_p\rightarrow B)$$

$$\Rightarrow\hat{\pi}\bar{\wedge}V\lesssim A_1\bar{\wedge}\cdots\bar{\wedge}V\lesssim A_m\bar{\wedge}W\lesssim A_{m+1}\bar{\wedge}\cdots\bar{\wedge}W\lesssim A_p\leqq W\lesssim B$$

is deducible in $\tilde{M}_{\mathrm{eff}}$.

*Proof.*

$$\text{(M0.5)}:\quad W \lesssim A_{m+1} \barwedge \cdots \barwedge W \lesssim A_p \leqq W \lesssim (A_{m+1} \wedge \cdots \wedge A_p) \qquad (1)$$

$$\text{(2.2)}:\quad \hat{\pi} \barwedge V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \leqq V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \qquad (2)$$

by (1), (2), (2.1), (2.2), (1.2):

$$\hat{\pi} \barwedge V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \barwedge W \lesssim A_{m+1} \barwedge \cdots \barwedge W \lesssim A_p$$

$$\leqq V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \barwedge W \lesssim (A_{m+1} \wedge \cdots \wedge A_p) \qquad (3)$$

premise $\Rightarrow V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \leqq V \lesssim (A_{m+1} \wedge \cdots \wedge A_p \to B)$ (R)

$\Rightarrow V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \leqq W \lesssim (A_{m+1} \wedge \cdots \wedge A_p \to B)$ (M0.11),(1.2)

$\Rightarrow V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \leqq W \lesssim (A_{m+1} \wedge \cdots \wedge A_p) \Rightarrow W \lesssim B)$ (M0.3)

$\Rightarrow V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \barwedge W \lesssim (A_{m+1} \wedge \cdots \wedge A_p) \leqq W \lesssim B$ (4.2′) (4)

$\Rightarrow$ proposition [by (3), (4), (1.2)] □

ad (2.5.2): Proposition: The rule

$$\underline{V} \leqq V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \Rightarrow V \lesssim B \Rightarrow \hat{\pi} \barwedge V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \leqq W \lesssim B$$

is deducible in $\tilde{M}_{\text{eff}}$.

*Proof.*

premise $\Rightarrow V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \leqq V \lesssim B$ (R)

$\Rightarrow \bar{\pi} \barwedge V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \leqq V \lesssim B$ (2.2), (1.2)

$\Rightarrow \bar{\pi} \barwedge V \lesssim A_1 \barwedge \cdots \barwedge V \lesssim A_m \leqq W \lesssim B$ (M0.11), (1.2) □

ad (2.5.3): Proposition: The rule

$$\underline{V} \leqq V \lesssim (A_{m+1} \wedge \cdots \wedge A_p \to B) \Rightarrow \hat{\pi} \barwedge W \lesssim A_{m+1} \barwedge \cdots \barwedge W \lesssim A_p \leqq W \lesssim B$$

is deducible in $\tilde{M}_{\text{eff}}$.

*Proof.*

$$\text{premise} \Rightarrow \underline{\mathrm{V}} \leqq W \lesssim (A_{m+1} \wedge \cdots \wedge A_p \rightarrow B) \qquad \text{(M0.11), (1.2)}$$

$$\Rightarrow \underline{\mathrm{V}} \leqq W \lesssim (A_{m+1} \wedge \cdots \wedge A_p) \Rightarrow W \lesssim B \qquad \text{(M0.7)}$$

$$\Rightarrow W \lesssim (A_{m+1} \wedge \cdots \wedge A_p) \leqq W \lesssim B \qquad \text{(R)}$$

$$\Rightarrow W \lesssim A_{m+1} \overline{\wedge} \cdots \overline{\wedge} W \lesssim A_p \leqq W \lesssim B \qquad \text{(M0.5), (1.2)}$$

$$\Rightarrow \text{proposition [by (2.2), (1.2)]} \qquad \square$$

(4) ad (5.3.1): Proposition: The rule

$$\underline{\mathrm{V}} \leqq V \lesssim (A_1 \wedge \cdots \wedge A_m \rightarrow \wedge) \Rightarrow \hat{\bar{\pi}} \overline{\wedge} V \lesssim A_1 \overline{\wedge} \cdots \overline{\wedge} V \lesssim A_m \leqq \overline{C}$$

is deducible in $\tilde{M}_{\mathrm{eff}}$ with any metaproposition $\overline{C}$.

*Proof.*

$$\text{premise} \Rightarrow \underline{\mathrm{V}} \leqq V \lesssim (A_1 \wedge \cdots \wedge A_m) \Rightarrow V \lesssim \Lambda \qquad \text{(M0.17)}$$

$$\Rightarrow V \lesssim (A_1 \wedge \cdots \wedge A_m) \leqq V \lesssim \Lambda \qquad \text{(R)}$$

$$\Rightarrow V \lesssim (A_1 \wedge \cdots \wedge A_m) \leqq W \lesssim \Lambda \qquad \text{(M0.11), (1.2)}$$

$$\Rightarrow V \lesssim (A_1 \wedge \cdots \wedge A_m) \leqq \overline{\Lambda} \qquad \text{(M0.1), (1.2)}$$

$$\Rightarrow V \lesssim (A_1 \wedge \cdots \wedge A_m) \leqq \overline{C} \qquad \text{definition of } \overline{\Lambda}\text{, (1.2)} \qquad (1)$$

$$\text{(2.2):} \quad \hat{\bar{\pi}} \overline{\wedge} V \lesssim A_1 \overline{\wedge} \cdots \overline{\wedge} V \lesssim A_m \leqq V \lesssim A_1 \overline{\wedge} \cdots \overline{\wedge} V \lesssim A_m$$

$$\Rightarrow \hat{\bar{\pi}} \overline{\wedge} V \lesssim A_1 \overline{\wedge} \cdots \overline{\wedge} V \lesssim A_m \leqq V \lesssim (A_1 \wedge \cdots \wedge A_m) \qquad \text{(M0.15), (1.2)}$$

$$\Rightarrow \text{proposition [by (1), (1.2)]} \qquad \square$$

ad (5.3.2): Proposition: The rule

$$\underline{\mathrm{V}} \leqq V \lesssim (A_{m+1} \wedge \cdots \wedge A_p \rightarrow \wedge) \Rightarrow \hat{\bar{\pi}} \overline{\wedge} W \lesssim A_{m+1} \overline{\wedge} \cdots \overline{\wedge} W \lesssim A_p \leqq \overline{C}$$

is deducible in $\tilde{M}_{\mathrm{eff}}$ with any metaproposition $\overline{C}$.

*Proof.*

$$\text{premise} \Rightarrow \underline{\mathrm{V}} \leqq W \lesssim (A_{m+1} \wedge \cdots \wedge A_p \to \Lambda) \qquad \text{(M0.11), (1.2)}$$

$$\Rightarrow \underline{\mathrm{V}} \leqq W \lesssim (A_{m+1} \wedge \cdots \wedge A_p) \Rightarrow W \lesssim \Lambda \qquad \text{(M0.7)}$$

$$\Rightarrow W \lesssim (A_{m+1} \wedge \cdots \wedge A_p) \leqq W \lesssim \Lambda \qquad \text{(R)}$$

$$\Rightarrow W \lesssim (A_{m+1} \wedge \cdots \wedge A_p) \leqq \overline{\Lambda} \qquad \text{(M0.1), (1.2)}$$

$$\Rightarrow W \lesssim (A_{m+1} \wedge \cdots \wedge A_p) \leqq \overline{C} \qquad \text{definition of } \overline{\wedge}\text{, (1.2)} \qquad (1)$$

$$\text{(2.2):} \quad \hat{\pi} \overline{\wedge} W \lesssim A_{m+1} \overline{\wedge} \cdots \overline{\wedge} W \lesssim A_p \leqq W \lesssim A_{m+1} \wedge \cdots \wedge W \lesssim A_p$$

$$\Rightarrow \hat{\pi} \overline{\wedge} W \lesssim A_{m+1} \overline{\wedge} \cdots \overline{\wedge} W \lesssim A_p \leqq W \lesssim (A_{m+1} \wedge \cdots \wedge A_p) \text{(M0.5), (1.2)}$$

$\Rightarrow$ proposition [by (1), (1.2)] □

(5) ad (6): Proposition: For any metaproposition $\overline{C}$ the figure $\hat{\pi} \overline{\wedge} W \lesssim \Lambda \leqq \overline{C}$ is deducible in $M_{\text{eff}}$.

*Proof.*

$$\text{(M0.1):} \ W \lesssim \Lambda \leqq \overline{\Lambda}$$

$$\Rightarrow \hat{\pi} \overline{\wedge} W \lesssim \Lambda \leqq \overline{\Lambda} \qquad \text{(2.2), (1.1)} \qquad (1)$$

$$\text{definition of } \overline{\Lambda}\text{:} \ \overline{\Lambda} \leqq \overline{C} \qquad (2)$$

By (1), (2), and (1.2) we get the proposition. □

(6) ad (k1.1), etc.: The rules transformed by $g_3$ are rules in $\tilde{M}_{\text{eff}}$, so they are deducible.

For the soundness proof consider the mapping

| | | | |
|---|---|---|---|
| $g_4$: | {figures of $\tilde{M}_{\text{eff}}$} | $\to$ | {tableaux of $\phi_{\text{eff}}(D_{sf}^m)$} |
| | $\overline{A} \leqq \overline{B}$ | $\to$ | $\overline{A} \Vert [\overline{B}]$ |
| | $\overline{A} \leqq \underline{\mathrm{V}}$ | $\to$ | $\overline{A} \Vert \overline{C} \Rightarrow \overline{C}$ |
| | $\overline{A} \leqq \overline{\Lambda}$ | $\to$ | $\overline{A} \Vert [\ \ ]$ |
| | $\underline{\mathrm{V}} \leqq \overline{A}$ | $\to$ | $\Vert \overline{A}$ |
| | $\overline{\Lambda} \leqq \overline{A}$ | $\to$ | $\overline{C} \overline{\wedge} \urcorner \overline{C} \Vert \overline{A}$ |

at which $\overline{C}$ is any metaproposition.

We have to show that the rules of $\tilde{M}_{\text{eff}}$ transformed by $g_4$ are deducible in $\phi_{\text{eff}}(D^m_{sf})$.

(1) ad (1.1)–(5.2): cf. Stachow (1978).

(2) ad (M0.1): $W \lesssim \bigwedge \| [\ \ ]$ is deducible in $\phi_{\text{eff}}(D^m_{sf})$ for it is a beginning [(6) with empty $\bar{\pi}$].

(3) ad (M0.3): Proposition: $W \lesssim (A \to B) \| [W \lesssim A \Rightarrow W \lesssim B]$ is deducible in $\phi_{\text{eff}}(D^m_{sf})$.

*Proof.*

$$(1.7):\quad \| V \lesssim [(A \to B) \wedge A] \to B$$

$$\Rightarrow \begin{array}{l} W \lesssim (A \to B) \\ W \lesssim A \end{array} \left\| \begin{array}{l} \\ W \lesssim A \end{array} \right. \qquad (2.5.3)$$

$$\Rightarrow \begin{array}{l} W \lesssim (A \to B) \\ W \lesssim A \end{array} \left\| \begin{array}{l} \\ [W \lesssim B] \end{array} \right. \qquad (5.1)$$

$$\Rightarrow W \lesssim (A \to B) \| W \lesssim A \Rightarrow W \lesssim B \qquad (2.3)$$

$$\Rightarrow \text{proposition } [\text{by } (5.1)]$$ □

(4) ad (M0.5): Proposition: $W \lesssim A \barwedge W \lesssim B \| [W \lesssim A \wedge B]$ is deducible in $\phi_{\text{eff}}(D^m_{sf})$.

*Proof.*

$$(1.2):\quad \| V \lesssim (A \wedge B \to A \wedge B)$$

$$(\bar{\pi} = W \lesssim A \barwedge W \lesssim B)$$

$$\Rightarrow \begin{array}{l} \bar{\pi} \\ W \lesssim A \\ W \lesssim B \end{array} \left\| \begin{array}{l} \\ \\ W \lesssim A \wedge B \end{array} \right. \qquad (2.5)$$

$$\Rightarrow \begin{array}{l} \bar{\pi} \\ W \lesssim A \\ W \lesssim B \end{array} \left\| \begin{array}{l} \\ \\ [W \lesssim A \wedge B] \end{array} \right. \qquad (5.1)$$

$$\Rightarrow \begin{array}{l} W \lesssim A \\ W \lesssim A \barwedge W \lesssim B \\ W \lesssim B \end{array} \left\| \begin{array}{l} \\ \\ [W \lesssim A \wedge B] \end{array} \right.$$

(Arguments within a column can be exchanged.)

$$\Rightarrow \begin{array}{l} W \lesssim A \\ W \lesssim A \barwedge W \lesssim B \end{array} \left\| \begin{array}{l} \\ [W \lesssim A \wedge B] \end{array} \right. \qquad (3.1.2)$$

$$\Rightarrow \begin{array}{l} W \lesssim A \barwedge W \lesssim B \\ W \lesssim A \end{array} \left\| \begin{array}{l} \\ [W \lesssim A \barwedge B] \end{array} \right.$$

$$\Rightarrow W \lesssim A \barwedge W \lesssim B \| [W \lesssim A \barwedge B] \qquad (3.1.1)$$ □

(5) ad (M0.11): Proposition: $V\lesssim A\,\|[W\lesssim A]$ is deducible in $\phi_{\text{eff}}(D^m_{sf})$.

*Proof.*

$$
\begin{array}{ll}
(1.1): \quad V\lesssim A\,\|\,V\lesssim A & \\
\Rightarrow V\lesssim A\,\|[V\lesssim A] & (5.1)\\
\Rightarrow \|\,V\lesssim A\Rightarrow V\lesssim A & (2.3)\\
\Rightarrow V\lesssim A\,\|\,W\lesssim A & (2.5.2)\\
\Rightarrow V\lesssim A\,\|[W\lesssim A] & (5.1)
\end{array}
$$

□

(6) ad (M0.13): Proposition: $V\lesssim(A\to B)\|[V\lesssim A\Rightarrow V\lesssim B]$ is deducible in $\phi_{\text{eff}}(D^m_{sf})$.

*Proof.*

$$
(1.1):\ \begin{array}{l} V\lesssim(A\to B)\\ V\lesssim A\end{array}\left\|\, V\lesssim A \right. \Rightarrow \begin{array}{l} V\lesssim(A\to B)\\ V\lesssim A\end{array}\left\|\, V\lesssim(V\to A)\right. \qquad (1)
$$

$$
(1.1):\ \begin{array}{l} V\lesssim(A\to B)\\ V\lesssim A\end{array}\left\|\, V\lesssim(A\to B)\right. \qquad (2)
$$

$$
(1),(2)\Rightarrow \begin{array}{l} V\lesssim(A\to B)\\ V\lesssim A\end{array}\left\|\, V\lesssim(V\to B)\right. \qquad (\text{k}1.2)
$$

$$
\Rightarrow \begin{array}{l} V\lesssim(A\to B)\\ V\lesssim A\end{array}\left\|\, V\lesssim B\right.
$$

etc. like in (3). □

(7) ad (M0.15): Proposition: $V\lesssim A\barwedge V\lesssim B\,\|[V\lesssim(A\wedge B)]$ is deducible in $\phi_{\text{eff}}(D^m_{sf})$.

*Proof.*

$$
(1.1):\quad \begin{array}{l} V\lesssim A\\ V\lesssim B\end{array}\left\|\, V\lesssim A\right. \quad \text{and} \quad \begin{array}{l} V\lesssim A\\ V\lesssim B\end{array}\left\|\, V\lesssim B\right.
$$

$$
\Rightarrow \quad \begin{array}{l} V\lesssim A\\ V\lesssim B\end{array}\left\|\, V\lesssim(V\to A)\right. \quad \text{and} \quad \begin{array}{l} V\lesssim A\\ V\lesssim B\end{array}\left\|\, V\lesssim(V\to B)\right.
$$

$$
\Rightarrow \quad \begin{array}{l} V\lesssim A\\ V\lesssim B\end{array}\left\|\, V\lesssim(V\to A\wedge B)\right. \qquad (\text{k}2.3)
$$

$$
\Rightarrow \quad \begin{array}{l} V\lesssim A\\ V\lesssim B\end{array}\left\|\, V\lesssim(A\wedge B)\right.
$$

etc. like in (4). □

(8) ad (M1.1): Proposition: $\|V\lesssim(A\to A)$ is deducible in $\phi_{\text{eff}}(D^m_{sf})$.

*Proof.* $\|V\lesssim(A\to A)$ is a beginning of $\phi_{\text{eff}}(D^m_{sf})$, viz., (k1.1) with empty $\bar{\pi}$. □

ad (M1.2): Proposition: The rule $\|V\lesssim(A\to B)$ and $\|V\lesssim(B\to C)$ $\Rightarrow \|V\lesssim(A\to C)$ is deducible in $\phi_{\text{eff}}(D^m_{sf})$.

*Proof.* This is a rule in $\phi_{\text{eff}}(D^m_{sf})$, viz., (k1.2) with empty $\bar{\pi}$. □

ad (M2.1)–(M5.3): like above with (k2.1), etc. ■

**5.2. The Calculus $M_{\text{eff}}(Q_{\text{eff}})$.** In order to get a modal calculus in the usual notation (Hughes, 1968) we introduce the symbol "Δ" instead of "$W\lesssim$" by which we want to emphasize that the respective figure is true for *any* $W$; the truth does not depend on a special $W_0$. Also "$V\lesssim$" is replaced by the symbol "$\vdash$" as an abbreviation for "$\vdash_{\bar{K}}$" with the calculus $K$ of the object language. (Remember that "$V\lesssim A$" was introduced as an abbreviation for $\vdash_{\overline{Q_{\text{eff}}}}$.)

The calculus $M_{\text{eff}}(Q_{\text{eff}})$ reads as follows:

(1.1)–(5.2) are the same rules like those in $\tilde{M}_{\text{eff}}$

(M0.1) $\Rightarrow \Delta\Lambda \leqq \bar{\Lambda}$

(M0.3) $\Rightarrow \Delta(A\to B) \leqq \Delta A \Rightarrow \Delta B$

(M0.5) $\Rightarrow \Delta A \bar{\wedge} \Delta B \leqq \Delta(A\wedge B)$

(M0.11) $\Rightarrow \vdash A \leqq \Delta A$

(M0.13) $\Rightarrow \vdash(A\to B) \leqq \vdash A \Rightarrow \vdash B$

(M0.15) $\Rightarrow \vdash A \bar{\wedge} \vdash B \leqq \vdash(A\wedge B)$

Furthermore we have again the beginnings and rules reflecting the calculus $\mathcal{K}$ or the object language, respectively, e.g.

(M1.1) $\Rightarrow \underline{V} \leqq \vdash(A\to A)$

(M1.2) $\underline{V} \leqq \vdash(A\to B)$ and $\underline{V} \leqq \vdash(B\to C) \Rightarrow \underline{V} \leqq \vdash(A\to C)$

*Theorem 5.2. Completeness and soundness of $M_{\text{eff}}(Q_{\text{eff}})$ with respect to $\tilde{M}_{\text{eff}}$.* The figure $\underline{V} \leqq \bar{A}^j(W\lesssim A_i, V\lesssim B_j)$ can be deduced in $\tilde{M}_{\text{eff}}$ iff the figure $\underline{V} \leqq \bar{A}^j(\Delta A_i, \vdash B_j)$ can be deduced in $M_{\text{eff}}(Q_{\text{eff}})$.

*Proof.* Because $W$ is a variable the place of which cannot be occupied by a fixed object proposition (as mentioned above) the following mapping $g_5$ is bijective:

$$g_5: \quad \{\text{figures of } \tilde{M}_{\text{eff}}\} \quad \to \quad \{\text{figures of } M_{\text{eff}}(Q_{\text{eff}})\}$$
$$\bar{A}^j(W \lesssim A_i, V \lesssim B_j) \quad \to \quad \bar{A}^j(\Delta A_i, \vdash B_j)$$

Therefore the completeness- and soundness proof is trivial. ■

*Theorem 5.3.* (a) The following figures and rules can be deduced in $M_{\text{eff}}(Q_{\text{eff}})$:

(R) $\underline{V} \leqq \bar{A} \Rightarrow \bar{B} \Leftrightarrow \bar{A} \leqq \bar{B}$

(M0.2) $\underline{V} \leqq \Delta V$

(M0.4) $\Delta A \underline{\vee} \Delta B \leqq \Delta(A \vee B)$

(M0.5′) $\Delta(A \wedge B) \leqq \Delta A \overline{\wedge} \Delta B$

(M0.6) $\Delta(\neg A) \leqq \daleth \Delta A$

(M0.7) $\underline{V} \leqq \Delta(A \to B) \Rightarrow \underline{V} \leqq \Delta A \Rightarrow \Delta B$

(M0.8) $\underline{V} \leqq \Delta A \underline{\vee} \Delta B \Rightarrow \underline{V} \leqq \Delta(A \vee B)$

(M0.9) $\underline{V} \leqq \Delta A \overline{\wedge} \Delta B \Leftrightarrow \underline{V} \leqq \Delta(A \wedge B)$

(M0.10) $\underline{V} \leqq \Delta(\neg A) \Rightarrow \underline{V} \leqq \daleth \Delta A$

(M0.11′) $\underline{V} \leqq \vdash A \Rightarrow \underline{V} \leqq \Delta A$

(M0.12) $\underline{V} \leqq \vdash V$

(b) The following rules are admissible in $M_{\text{eff}}(Q_{\text{eff}})$:

(Z1) $\underline{V} \leqq \Delta A \Rightarrow V \leqq \vdash A$

(Z2) $\underline{V} \leqq (\Delta A \Rightarrow \Delta B) \Rightarrow V \leqq \Delta(A \to B)$

(A) $\underline{V} \leqq (\Delta A \overline{\wedge} \Delta B \Rightarrow \Delta C) \Leftrightarrow \underline{V} \leqq \vdash(A \wedge B \to C)$

(A) is the formulation of the "theorem of Aristotle" within the metalogical calculus $M_{\text{eff}}(Q_{\text{eff}})$. A proof of the above assertions will not be given here. Burghardt (1979).

The first part of the calculus $M_{\text{eff}}(Q_{\text{eff}})$ consists of the rules (1.1)–(5.2) that establish the *intuitionistic* calculus of *formal metalogic* already mentioned by Mittelstaedt (1979). Within this calculus all metapropositions can be deduced which are true without reference to the object language, e.g., $\Delta A \overline{\wedge} \daleth \vdash B \lessdot \Delta A \underline{\vee} \Delta C$. It is obvious that the formal metalogic is invariant if one changes the object language. Another part of $M_{\text{eff}}(Q_{\text{eff}})$

covers the rules (M1.1)–(M5.3) reflecting the object language. By means of this part a deduction of a figure in the calculus of the object language can be performed on the level of the metalanguage. If we change the object language, of course, the respective rules of (M1.1)–(M5.3) will be changed in the same way too. For example, taking the intuitionistic logic $L_{\text{eff}}$ as representing our object language the rules (M4.2)–(M4.4) are to be replaced by

$$(\text{M4.2*})\quad \underline{V} \leqq \vdash (A \wedge B \to C) \Rightarrow \underline{V} \leqq \vdash (B \to (A \to C))$$

and instead of (M5.2)–(M5.3) we have

$$(\text{M5.2*})\quad \underline{V} \leqq \vdash (A \wedge B \to \Lambda) \Rightarrow \underline{V} \leqq \vdash (B \to \neg A)$$

Using $Q_{\text{eff}}^{(f)}$ instead of $Q_{\text{eff}}$ as the object language the *factual beginnings* would be taken into account by the new beginnings $\underline{V} \leqq \vdash (A_i \to B_i)$ in $M_{\text{eff}}(Q_{\text{eff}}^{(f)})$.

The third part of $M_{\text{eff}}(Q_{\text{eff}})$ is called the *modal part* and contains the remaining rules (M0.1), (M0.3), (M0.5), (M0.11), (M0.13), and (M0.15), which establish the connection between the part representing the object language and the formal metalogic. Only by means of the modal part is it possible to use the knowledge of the object language in the deduction of a (nonelementary) metaproposition. Using the deduced rule (M0.11) and the definition of $\overline{\Lambda}$ the first rule of the modal part, viz., (M0.1), yields the logical equivalence of the "metafalsum" $\overline{\Lambda}$ and the "object falsum" $\Lambda$. Analogously the equivalence of the "metaverum" $\underline{V}$ and the "objectverum" $V$ is established by (M0.2) and (M0.12).

Finally let us prove an interesting result concerning the variation of the modal part if the object language is changed. We will concentrate on the calculi $K$ (representing the object language) founded by a dialogic semantics: $Q_{\text{eff}}$ (effective quantum logic), $Q$ [full quantum logic (Mittelstaedt, 1978)], $L_{\text{eff}}$ (intuitionistic logic), $L$ (classical logic), and the respective extensions $Q_{\text{eff}}^{(f)}$, $Q^{(f)}$, $L_{\text{eff}}^{(f)}$, $L^{(f)}$ including the *factual beginnings*. The mutual relations of these calculi form a Boolean lattice, as shown in Figure 1. The lattice relation $K_1 \leqslant K_2$, e.g., $Q_{\text{eff}} \leqslant L_{\text{eff}}$, means that every figure deducible in $K_1$ is also deducible in $K_2$.

*Theorem 5.4. Invariance of the modal part of* $M_{\text{eff}}(Q_{\text{eff}})$. For every calculus $K$ of the above-mentioned calculi with a dialogic foundation the modal part of the respective metalogical calculus $M_{\text{eff}}(K)$ is the same as the one in $M_{\text{eff}}(Q_{\text{eff}})$.

*Proof.* First it is necessary to clarify one special aspect. Of course if we change the object language we would be able to change in an appropriate manner only that part of $M_{\text{eff}}(Q_{\text{eff}})$ reflecting the object language and to

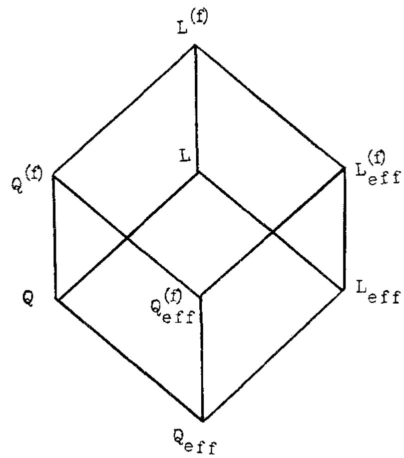


**Fig. 1**

say, "Well, this is our new metalogic." Doing so we would construct the metacalculus $M_{\text{eff}}(K)$ *ad hoc*. But this is not the way to get a logical calculus we are interested in! We are interested in calculi with a dialogic semantics, and therefore we have to ask, "If we consider a calculus $K$ representing the object language what changes have to be made within the rules of the metadialog and what conclusions these changes yield with respect to the calculus $M_{\text{eff}}(K)$?"

(i) Changes of the rules of $D_s$: The frame rules and the argument rules $A_f(1)$ and $A_f(3)$ are independent of the fact that we consider a metalanguage. $A_s(2)$ depends on the object language only as far as in part bb) the deductions are performed in $Q_{\text{eff}}$. Using another calculus $K$ instead of $Q_{\text{eff}}$ the rule $A_s(2)$ is preserved except that the deductions have to be made by means of $K$. Of course we assume that $V \leqslant \Lambda$ is not deducible in $K$. Therefore the rule $A_s(4)$ is preserved too except that again the deductions must be performed by means of $K$.

(ii) Changes of the rules of $D_{sf}^m$: $A^m(5)$ was founded outside of the dialog, and therefore this rule is preserved if we change the object language. For the transition from $A_s(2)$ to $A_{sf}^m(2)$ we needed the "generalized theorem of Aristotle," the proof of which only uses rules of $Q_{\text{eff}}$ corresponding to lattice properties (cf. Mittelstaedt, 1978). These rules also appear in the calculi $K$ mentioned above, and the generalized theorem of Aristotle can be proved for these calculi. The transition from $A_s(4)$ to $A_{sf}^m(4)$ needed a lemma proved by using the completeness and soundness of $Q_{\text{eff}}$ with respect to a dialogic semantics. But the calculi $K$ mentioned above have such a dialogic semantics too and a respective lemma can be proved.

So we have the following result: If we change the object language the rules of $D_{sf}^m$ are modified only as far as one has to change $\mathcal{K}$, respectively.

(iii) Changes in the proofs of the Theorems 4.1, 4.2, 5.1, and 5.2: In the completeness- and soundness proof of the tableaux calculus the calculus $K$ (or $\mathcal{K}$, respectively) of the object language only has been used for establishing the tableaux rules (1.2)–(1.10) and (4.0)–(4.9) which reflect $K$ equivalently. Concerning $\tilde{M}_{\text{eff}}$ we must distinguish completeness and soundness:

($\alpha$) In order to get a calculus $\tilde{M}_{\text{eff}}$ complete with respect to $\phi_{\text{eff}}(D_{sf}^m)$ and with respect to $K$ (instead of $Q_{\text{eff}}$) it is not necessary to change the modal part of $\tilde{M}_{\text{eff}}$. For we need the object language only for establishing the rules (1.1)–(M5.3).

($\beta$) In the soundness proof of the modal part of $\tilde{M}_{\text{eff}}$ only some rules of the tableaux calculus were needed which reflect the transitivity, the supremum property, and the properties of $V$ which are also valid within the above mentioned calculi $K$. Of course the transition from $\tilde{M}_{\text{eff}}$ to $M_{\text{eff}}(Q_{\text{eff}})$ or $M_{\text{eff}}(K)$ is only a formal one and is therefore independent of the object language.

## ACKNOWLEDGEMENT

I wish to thank Professor P. Mittelstaedt and Dr. E.-W. Stachow for many helpful discussions during my work as a doctoral student at the Institut für Theoretische Physik der Universität zu Köln, West Germany.

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